

# NON-GAUSSIAN EFFECTS IN THE COSMIC MICROWAVE BACKGROUND FROM INFLATION

Alejandro Gangui\*

*SISSA – International School for Advanced Studies,  
via Beirut 2 – 4, 34013 Trieste, Italy*

To appear in *Physical Review D*

## Abstract

The presence of non-Gaussian features in the Cosmic Microwave Background (CMB) radiation maps represents one of the most long-awaited clues in the search for the actual structure of the primordial radiation, still needing confirmation. These features could shed some light on the non trivial task of distinguishing the real source of the primeval perturbations leading to large scale structure. One of the simplest non-Gaussian signals to search is the (dimensionless) skewness  $\mathcal{S}$ . Explicit computations for  $\mathcal{S}$  are presented in the frame of physically motivated inflationary models (natural, intermediate and polynomial potential inflation) in the hope of finding values in agreement with estimated quantities from large angle scale (e.g., *COBE* DMR) maps. In all the cases considered the non-Gaussian effects turn out to lie below the level of theoretical uncertainty (cosmic variance). The possibility of unveiling the signal for  $\mathcal{S}$  with multiple-field models is also discussed.

PACS number(s): 98.80.Cq, 98.80.Es, 98.70.Vc, 98.80.-k

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\*Electronic mail: [gangui@tsmi19.sissa.it](mailto:gangui@tsmi19.sissa.it)

# 1 Introduction

The existence of anisotropies in the CMB radiation as recently detected by *COBE* [1] and subsequently confirmed both by balloon–borne scans at shorter wavelength [2] and by ground–based intermediate angular scale observations [3] has triggered a large body of literature dealing with the non trivial task of finding the correct statistics able to disentangle the relevant information out of the primeval radiation maps. Both small and large angle scale probes of the microwave sky have been and are the real data our theoretical models must reproduce before we might call them *viable*. Discrimination between the two main theories for the origin of primordial perturbations, namely, whether these are due to topological defects [4] produced during a GUT phase transition or to early inflationary quantum fluctuations [5], has by now become a difficult matter.

In this paper we will work in the frame of inflation and, in particular, we will be mainly concerned with single–field inflationary potentials. Some of the most popular models are characterized by their simplicity and universality (such as quadratic and quartic chaotic potentials), by their being exact solutions of the equations of motion for the inflaton (like power–law and intermediate inflation) or by their particle physics motivation (as natural inflation with an axion–like potential). In contrast with these simpler models where one has just one relevant parameter more general potentials, with more freedom, were also considered in the literature. One example of this is the polynomial potential [6] which for an adequate choice of the parameters was found to lead to broken scale invariant spectra on a wide range of scales with interesting consequences for large scale structure.

One should also worry about initial conditions [7]. While for single–field models the only effect of kinetic terms consists in slightly changing the initial value of  $\phi$  (leaving invariant the phase space of initial field values leading to sufficient inflation), for models with more than one scalar field initial conditions can be important, e.g., for double inflation [8] (with two stages of inflation each one dominated by a different inflaton field) leading to primordial non–Gaussian perturbations on cosmological interesting scales. Within the latter models,

however, the question of how probable it is that a certain initial configuration will be realized in our neighbouring universe should be addressed. More recently other examples of interesting multiple-field models with broken scale invariance have also been considered (see e.g. Ref.[9]). Here all scalar fields contribute to the energy density and non-Gaussian features are produced when the scalar fields pass over the interfaces of continuity of the potential. Extension of the single-field stochastic approach developed in Ref.[10] for calculating the CMB angular bispectrum generated through Sachs–Wolfe effect from primordial curvature perturbations in the inflaton in order to include many scalar fields is therefore needed [11].

The plan of the paper is as follows. In Section 2 a brief overview of some general results is given while in Section 3 we concentrate on trying to extract numerical values for the non-Gaussian signal (the dimensionless skewness in this case) predicted in the frame of three different inflationary models. Section 4 contains some general conclusions.

## 2 The CMB skewness

We will here briefly summarize the steps that lead to the calculation of the mean two- and three-point functions of the temperature anisotropies and in particular to the zero-lag limit of the latter, namely the skewness.

As usual we expand the temperature fluctuation in spherical harmonics  $\frac{\Delta T}{T}(\vec{x}; \hat{\gamma}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell}^m(\vec{x}) \mathcal{W}_{\ell} Y_{\ell}^m(\hat{\gamma})$ , where  $\vec{x}$  specifies the position of the observer and the unit vector  $\hat{\gamma}$  points in a given direction from  $\vec{x}$ .  $\mathcal{W}_{\ell}$  represents the window function of the specific experiment. Setting  $\mathcal{W}_0 = \mathcal{W}_1 = 0$  automatically accounts for both monopole and dipole subtraction; for  $\ell \geq 2$  one can take  $\mathcal{W}_{\ell} \simeq \exp\left[-\frac{1}{2}\ell(\ell+1)\sigma^2\right]$ , where  $\sigma$  is the dispersion of the antenna-beam profile, which measures the angular response of the detector (e.g. [12]). In some cases the quadrupole term is also subtracted from the maps (e.g. [1]); in this case we also set  $\mathcal{W}_2 = 0$ . The multipole coefficients  $a_{\ell}^m$  are here considered as zero-mean non-Gaussian random variables whose statistics derives from that of the gravitational potential through the Sachs–Wolfe relation  $\frac{\Delta T}{T}(\vec{x}; \hat{\gamma}) = \frac{1}{3}\Phi(\vec{x} + r_0 \hat{\gamma})$ , where  $r_0 = 2/H_0$  is the horizon

distance and  $H_0$  the Hubble constant.

In the frame of the inflationary model, the calculations reported in Ref.[10] lead to general expressions for the mean two- and three-point functions of the primordial gravitational potential, namely  $\langle \Phi(r_0 \hat{\gamma}_1) \Phi(r_0 \hat{\gamma}_2) \rangle = (9\mathcal{Q}^2/20\pi) \sum_{\ell \geq 0} (2\ell + 1) P_\ell(\hat{\gamma}_1 \cdot \hat{\gamma}_2) \mathcal{C}_\ell$  and

$$\begin{aligned} \langle \Phi(r_0 \hat{\gamma}_1) \Phi(r_0 \hat{\gamma}_2) \Phi(r_0 \hat{\gamma}_3) \rangle &= \frac{81\pi^2 \mathcal{Q}^4}{25(2\pi)^4} \Phi_3 \sum_{j, \ell \geq 0} (2j + 1)(2\ell + 1) \mathcal{C}_j \mathcal{C}_\ell \\ &\times [P_j(\hat{\gamma}_1 \cdot \hat{\gamma}_3) P_\ell(\hat{\gamma}_1 \cdot \hat{\gamma}_2) + P_j(\hat{\gamma}_2 \cdot \hat{\gamma}_1) P_\ell(\hat{\gamma}_2 \cdot \hat{\gamma}_3) + P_j(\hat{\gamma}_3 \cdot \hat{\gamma}_1) P_\ell(\hat{\gamma}_3 \cdot \hat{\gamma}_2)] , \end{aligned} \quad (1)$$

with  $P_\ell$  a Legendre polynomial and where  $\Phi_3$  is a model-dependent coefficient. These expectation values are a statistical average over the ensemble of possible observers and can only depend upon the needed number of angular separations.

The  $\ell$ -dependent coefficients  $\mathcal{C}_\ell$  are defined by  $\langle Q_\ell^2 \rangle \equiv \frac{(2\ell+1)}{5} \mathcal{Q}^2 \mathcal{C}_\ell$ , with  $\mathcal{Q} = \langle Q_2^2 \rangle^{1/2}$  the *rms* quadrupole, and are related to the gravitational potential power-spectrum  $P_\Phi(k)$  through  $\mathcal{C}_\ell = \int_0^\infty dk k^2 P_\Phi(k) j_\ell^2(kr_0) / \int_0^\infty dk k^2 P_\Phi(k) j_2^2(kr_0)$ , where  $j_\ell$  is the  $\ell$ -th order spherical Bessel function. The *rms* quadrupole is simply related to the quantity  $Q_{rms-PS}$  defined in Ref.[1, 13]:  $\mathcal{Q} = \sqrt{4\pi} Q_{rms-PS} / T_0$ , with mean temperature  $T_0 = 2.726 \pm 0.01K$  [14]. For the scales of interest we can make the approximation  $P_\Phi(k) \propto k^{n-4}$ , where  $n$  corresponds to the primordial index of density fluctuations (e.g.  $n = 1$  is the Zel'dovich, scale-invariant case), in which case [15, 16] we have  $\mathcal{C}_\ell = \Gamma\left(\ell + \frac{n}{2} - \frac{1}{2}\right) \Gamma\left(\frac{9}{2} - \frac{n}{2}\right) / \left(\Gamma\left(\ell + \frac{5}{2} - \frac{n}{2}\right) \Gamma\left(\frac{3}{2} + \frac{n}{2}\right)\right)$ . The equations above allow us to compute the angular spectrum,  $\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2*} \rangle = \delta_{\ell_1 \ell_2} \delta_{m_1 m_2} \mathcal{Q}^2 \mathcal{C}_{\ell_1} / 5$ , and the angular bispectrum,

$$\langle a_{\ell_1}^{m_1} a_{\ell_2}^{m_2} a_{\ell_3}^{m_3*} \rangle = \frac{3\mathcal{Q}^4}{25} \Phi_3 [\mathcal{C}_{\ell_1} \mathcal{C}_{\ell_2} + \mathcal{C}_{\ell_2} \mathcal{C}_{\ell_3} + \mathcal{C}_{\ell_3} \mathcal{C}_{\ell_1}] \mathcal{H}_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} , \quad (2)$$

where the coefficients  $\mathcal{H}_{\ell_1 \ell_2 \ell_3}^{m_1 m_2 m_3} \equiv \int d\Omega_{\hat{\gamma}} Y_{\ell_1}^{m_1*}(\hat{\gamma}) Y_{\ell_2}^{m_2}(\hat{\gamma}) Y_{\ell_3}^{m_3}(\hat{\gamma})$  are only non-zero if the indices  $\ell_i$ ,  $m_i$  ( $i = 1, 2, 3$ ) fulfill the relations:  $|\ell_j - \ell_k| \leq \ell_i \leq |\ell_j + \ell_k|$ ,  $\ell_1 + \ell_2 + \ell_3 = even$  and  $m_1 = m_2 + m_3$ . From the last equation we obtain the general form of the mean three-point correlation function for the temperature perturbations whose explicit form we will not need in the present work. Some simplifications occur for the CMB mean skewness  $\langle C_3(0) \rangle$  for

which we obtain

$$\begin{aligned} \langle C_3(0) \rangle &= \frac{3\mathcal{Q}^4}{25(4\pi)^2} \Phi_3 \sum_{\ell_1, \ell_2, \ell_3} (2\ell_1 + 1)(2\ell_2 + 1)(2\ell_3 + 1) [\mathcal{C}_{\ell_1} \mathcal{C}_{\ell_2} + \mathcal{C}_{\ell_2} \mathcal{C}_{\ell_3} + \mathcal{C}_{\ell_3} \mathcal{C}_{\ell_1}] \\ &\times \mathcal{W}_{\ell_1} \mathcal{W}_{\ell_2} \mathcal{W}_{\ell_3} \mathcal{F}_{\ell_1 \ell_2 \ell_3} \end{aligned} \quad (3)$$

where the coefficients  $\mathcal{F}_{\ell_1 \ell_2 \ell_3} \equiv (4\pi)^{-2} \int d\Omega_{\hat{\gamma}} \int d\Omega_{\hat{\gamma}'} P_{\ell_1}(\hat{\gamma} \cdot \hat{\gamma}') P_{\ell_2}(\hat{\gamma} \cdot \hat{\gamma}') P_{\ell_3}(\hat{\gamma} \cdot \hat{\gamma}')$  may be suitably expressed in terms of products of factorials of  $\ell_1$ ,  $\ell_2$  and  $\ell_3$ , using standard relations for Clebsch–Gordan coefficients.

By working in the frame of the stochastic approach to inflation [17, 18] we are able to compute the three–point function for the inflaton field perturbation  $\delta\phi$  [10]. Among the primordial perturbation scales we find those that stretched to super–horizon sizes approximately 60 e–foldings before the end of inflationary era to re–enter the Hubble radius next, during matter domination, as energy–density perturbations (and thus affecting also the gravitational potential  $\Phi$ ). During decoupling these scales are still greater than the horizon and therefore no microphysics alters the primevally imprinted signal they carry. The absence of non–linear evolution for these large angle scales makes physics simple and therefore highly predictive.

The stochastic analysis naturally takes into account all the multiplicative effects in the inflaton dynamics that are responsible for the non–Gaussian features. Extra–contributions to the three–point function of the gravitational potential also arise as a consequence of the non–linear relation between  $\Phi$  and  $\delta\phi$  [19]. When all primordial second–order effects are taken into account we get

$$\mathcal{Q}^2 = \frac{8\pi^2 H_{60}^2}{5m_P^2 X_{60}^2} \frac{\Gamma(3-n)\Gamma\left(\frac{3}{2} + \frac{n}{2}\right)}{\left[\Gamma\left(2 - \frac{n}{2}\right)\right]^2 \Gamma\left(\frac{9}{2} - \frac{n}{2}\right)} \quad (4)$$

for the *rms* quadrupole, while for the “dimensionless” skewness  $\mathcal{S} \equiv \langle C_3(0) \rangle / \langle C_2(0) \rangle^{3/2}$  we find [20]

$$\mathcal{S} = \frac{\sqrt{45\pi}}{32\pi^2} \mathcal{Q} \left[ X_{60}^2 - 4m_P X_{60}' \right] \mathcal{I}(n) \quad (5)$$

where we denoted with  $X_{60}$  the value of the steepness of the potential  $X(\phi) = m_P V'(\phi)/V(\phi)$  evaluated at  $\phi_{60}$  (the value of the inflaton 60 e-foldings before the end of inflation) and where  $\mathcal{I}(n)$  is a spectral index-dependent geometrical factor of order unity for interesting values of  $n$ . [10]

It is also of interest here to calculate the accurate form of the spectrum of primordial perturbations (e.g. by finding the value of the spectral index) at the moment the scales relevant for our study left the Hubble radius. Our expression  $\mathcal{S}$  for the dimensionless skewness is accurate up to second order in perturbation theory (the first non vanishing order) and therefore we should calculate  $n$  at least to the same order. We will borrow the notation for the slow-roll expansion parameters from Ref.[21]. These are defined as  $\epsilon(\phi) = m_P^2 (H'(\phi)/H(\phi))^2 / 4\pi$ ,  $\eta(\phi) = m_P^2 H''(\phi)/(4\pi H(\phi))$  and  $\xi(\phi) = m_P^2 H'''(\phi)/(4\pi H'(\phi))$  [22]. In the slow-roll approximation  $\epsilon$  and  $\eta$  are less than one. The same is not true in general for  $\xi$  and this may cause consistency problems when this term is incorrectly neglected (see the discussion in [21]). We will see below examples of this.

Let  $\mathcal{Q}_S^2$  ( $\mathcal{Q}_T^2$ ) be the contribution of the scalar (tensor) perturbation to the variance of the quadrupole temperature anisotropy. The complete second-order expressions for the tensor to scalar ratio and for the spectral index are given respectively by  $R \equiv \mathcal{Q}_T^2/\mathcal{Q}_S^2 \simeq 14\epsilon[1 - 2C(\eta - \epsilon)]$  and

$$n = 1 - 2\epsilon \left[ 2 - \frac{\eta}{\epsilon} + 4(C + 1)\epsilon - (5C + 3)\eta + C\xi(1 + 2(C + 1)\epsilon - C\eta) \right], \quad (6)$$

evaluated at  $\phi \simeq \phi_{60}$ . In this equation  $C \equiv -2 + \ln 2 + \gamma \simeq -0.7296$  and  $\gamma = 0.577$  is the Euler–Mascheroni constant [23].

It was realized [24, 25, 26] that a positive detection of a non-zero three-point function (an in particular a non-zero skewness) in the temperature fluctuations on the microwave sky *does not* imply an intrinsically non-Gaussian underlying field. This problem is related to the so-called “cosmic variance”. Relevant for large angular scale fluctuations, this variance is in fact the strongest source of (theoretical) noise we have to deal with and essentially reflects the impossibility of making observations in more than one universe. One way to quantify this

effect is through the *rms* skewness of a Gaussian field  $\langle C_3^2(0) \rangle_{Gauss}^{1/2}$ , which we may express as  $\langle C_3^2(0) \rangle_{Gauss} = 3 \int_{-1}^1 d\cos\alpha \langle C_2(\alpha) \rangle^3$ . A convenient quantity to compare with  $\mathcal{S}$  is the normalized *rms* skewness  $\langle C_3^2(0) \rangle_{Gauss}^{1/2} / \langle C_2(0) \rangle^{3/2}$ . For interesting values of  $n$  this ratio is of the order  $\sim 0.1$ . A rough criterion for the feasibility of detecting primordial non-Gaussian signatures could be expressed as  $\langle C_3(0) \rangle \gtrsim \langle C_3^2(0) \rangle_{Gauss}^{1/2}$ . Unfortunately as we will see for the models we work with here this is far from being the case; as a result primordial features cannot emerge. It is worth mentioning that recent analyses of the three-point function and the skewness from *COBE* data [27, 28] are also consistent with quasi-Gaussian fluctuations.

Now let us turn to the examples.

## 3 Worked Examples

### 3.1 Natural Inflation

To begin with let us consider the Natural inflationary scenario. First introduced in [29] this model borrows speculative ideas from axion particle physics [30]. Here the existence of disparate mass scales leads to the explanation of why it is physically attainable to have potentials with a height many orders of magnitude below its width [31], as required for successful inflation where usually self coupling constants are fine-tuned to very small values.

This model considers a Nambu-Goldstone (N-G) boson, as arising from a spontaneous symmetry breakdown of a global symmetry at energy scale  $f \sim m_P$ , playing the role of the inflaton. Assuming there is an additional explicit symmetry breaking phase at mass scale  $\Lambda \sim m_{GUT}$  these particles become pseudo N-B bosons and a periodic potential due to instanton effects arises. The simple potential (for temperatures  $T \leq \Lambda$ ) is of the form  $V(\phi) = \Lambda^4[1 + \cos(\phi/f)]$ .

For us this  $V(\phi)$  constitutes an axion-like model with the scales  $f$  and  $\Lambda$  as free parameters. It is convenient to split the parameter space into two regions. In the  $f \gg m_P$  zone the whole inflationary period happens in the neighbourhood of the minimum of the potential, as

may be clearly seen from the slow-rolling equation [32]  $|\cos(\phi/f)| \leq 24\pi/(24\pi + (m_P/f)^2)$  which is only violated near  $\phi_{end} \simeq \pi f$ , and from the small value of the steepness  $|X| = \frac{m_P}{f} \tan(\frac{\phi}{2f})$  for  $\phi$  smaller than  $\phi_{end}$  [33]. Thus by expanding  $V(\phi)$  around the minimum it is easy to see the equivalence between this potential and the quadratic one  $V \sim m^2(\phi - \pi f)^2/2$  with  $m^2 = \Lambda^4/f^2$ . We have already studied the latter in Ref.[10] and we will add nothing else here. On the other hand, let us consider the other regime, where  $f \lesssim m_P$ . Reheating temperature considerations place a lower limit on the width  $f$  of the potential. For typical values of the model parameters involved, a temperature  $T_{RH} \lesssim 10^8 \text{GeV}$  is attained. GUT baryogenesis via the usual out-of-equilibrium decay of X-bosons necessitates instead roughly  $T_{RH} \sim 10^{14} \text{GeV}$  (the mass of the gauge bosons) for successful reheating [32]. Thus the final temperature is not high enough to create them from the thermal bath. Baryon-violating decays of the field and its products could be an alternative to generate the observed asymmetry if taking place at  $T_{RH} > 100 \text{GeV}$ , the electroweak scale. This yields the constraint  $f \gtrsim 0.3m_P$  [34], implying  $n \gtrsim 0.6$  (see below). Attractive features of this model include the possibility of having a density fluctuation spectrum with extra power on large scales. Actually for  $f \lesssim 0.75 m_P$  the spectral index may be accurately expressed as  $n \simeq 1 - m_P^2/8\pi f^2$  [35]. This tilt in the spectrum as well as the negligible gravitational wave mode contribution to the CMB anisotropy might lead to important implications for large scale structure.[36]

Let us take now  $f \simeq 0.446 m_P$  corresponding to  $n \simeq 0.8$ . Slow-rolling requirements are satisfied provided the accelerated expansion ends by  $\phi_{end} \simeq 2.78f$ , very near the minimum of the potential. Furthermore, the slow-rolling solution of the field equations yields the value of the scalar field 60 e-foldings before the end of inflation  $\sin(\phi_{60}/2f) \simeq \exp(-15m_P^2/4\pi f^2)$  where we approximated  $\phi_{end} \simeq \pi f$ .

We find  $X_{60}^2 - 4m_P X'_{60} = (m_P/f)^2[2 + \sin^2(\phi_{60}/2f)](1 - \sin^2(\phi_{60}/2f))^{-1} \simeq 2(m_P/f)^2$ .

Two years of data by *COBE* [13] are not yet enough to separately constrain the amplitude of the quadrupole and the spectral index. A maximum likelihood analysis yields  $Q_{rms-PS} = 17.6 \exp[0.58(1 - n)] \mu K$ . By making use of Eq.(4) for the *rms* quadrupole, *COBE* results

constrain the value of the free parameter  $\Lambda \simeq 1.41 \times 10^{-4} m_P$ . We find  $\mathcal{S} \simeq 3.9 \times 10^{-5}$ , a rather small signal for the non-Gaussian amplitude of the fluctuations.

### 3.2 Intermediate Inflation

We will now study a class of universe models where the scale factor increases at a rate intermediate between power-law inflation –as produced by a scalar field with exponential potential [37] – and the standard de Sitter inflation. In Ref.[38] Barrow shows that it is possible to parameterize these solutions by an equation of state with pressure  $p$  and energy density  $\rho$  related by  $\rho + p = \gamma \rho^\lambda$ , with  $\gamma$  and  $\lambda$  constants. The standard perfect fluid relation is recovered for  $\lambda = 1$  leading to the  $a(t) \sim e^{H_{inf}t}$  ( $H_{inf}$  constant during inflation) solution of the dynamical equations when the spatial curvature  $k = 0$  and  $\gamma = 0$ , while  $a(t) \sim t^{2/3\gamma}$  for  $0 < \gamma < 2/3$ . This non-linear equation of state (and consequently the two limiting accelerated expansion behaviours) can be derived from a scalar field with potential  $V = V_0 \exp(-\sqrt{3\gamma}\phi)$ . On the other hand for  $\lambda > 1$  we have  $a(t) \propto \exp(At^f)$  (intermediate inflation) with  $A > 0$  and  $0 < f \equiv 2(1 - \lambda)/(1 - 2\lambda) < 1$  and again in this last case it is possible to mimic the matter source with that produced by a scalar field  $\phi$ , this time with potential

$$V(\phi) = \frac{8A^2}{(\beta + 4)^2} \left( \frac{m_P}{\sqrt{8\pi}} \right)^{2+\beta} \left( \frac{\phi}{\sqrt{2A\beta}} \right)^{-\beta} \left( 6 - \frac{\beta^2 m_P^2}{8\pi} \phi^{-2} \right) \quad (7)$$

with  $\beta = 4(f^{-1} - 1)$ .

The equations of motion for the field in a  $k = 0$  Friedmann universe may be expressed by  $3H^2 = 8\pi(V(\phi) + \dot{\phi}^2/2)/m_P^2$  and  $\ddot{\phi} + 3H\dot{\phi} = -V'$ . Exact solutions for these equations with the potential of Eq.(7) are of the form [38, 39]:  $H(\phi) = Af(A\beta/4\pi)^{\beta/4}(\phi/m_P)^{-\beta/2}$  and  $\phi(t) = (A\beta/4\pi)^{1/2}t^{f/2}m_P$ . Solutions are found for all  $\phi > 0$  but only for  $\phi^2 > (\beta^2/16\pi)m_P^2$  we get  $\ddot{a} > 0$  (i.e. inflation). In addition  $\beta > 1$  is required to ensure that the accelerated expansion occurs while the scalar field rolls (not necessarily slowly) down the potential, in the region to the right of the maximum (as it is generally the case).

From the full potential (7) we may compute the value of the dimensionless skewness  $\mathcal{S}$  (for convenience we will be taking the field  $\phi$  normalized in Planck mass units from here on)

$$\mathcal{S} = 0.17\mathcal{Q} \left[ \frac{\beta(\beta-4)}{\phi^2} + \frac{4\beta^4(\beta-1) - 192\pi\beta^2(\beta-6)\phi^2}{\phi^2(48\pi\phi^2 - \beta^2)^2} \right], \quad (8)$$

evaluated at  $\phi \simeq \phi_{60}$ . In Eq.(8) we took  $\mathcal{I}(n) \simeq 4.5$  (from Eq.(5)), that is the case for the specific examples we discuss below. A plot of this quantity as a function of  $d \equiv \phi^2 - \beta^2/16\pi > 0$  (the value of the squared of the field beyond the minimum allowed) for different values of  $\beta$  is given in Fig. 1. Note that both positive and negative values of  $\mathcal{S}$  are therefore allowed just by modifying the choice of  $\beta$ . Another generic feature is the rapid decrease of the non-Gaussian amplitude for increasing values of the field beyond  $\beta/\sqrt{16\pi}$  (i.e.  $d > 0$ ). Clearly this is because for large  $\phi$  we approach the slow-roll region where the steepness becomes increasingly small.

Similar calculations may be done for the spectral index. The explicit expression of  $n$  calculated from our potential (7) is complicated and not very illuminating. Fig. 2 illustrates the variation of  $n$  as a function of  $d$  (i.e. the scale dependence of the spectral index) for different values of the parameter  $\beta$ .

These two figures show that for acceptable values of the spectral index very small amplitudes for  $\mathcal{S}$  are generally predicted. As an example we consider  $\beta = 1.2$ . This ansatz yields a negative  $\mathcal{S}$  being  $\phi_{60} \simeq 0.37$  the value of the field that maximizes  $|\mathcal{S}| \simeq 2.25\mathcal{Q}$ . We show in Fig. 3 the form of the inflaton potential (7) for this particular  $\beta$ . We see that for the scales that exit the Hubble radius 60 e-foldings before the end of inflation [40] the value of the inflaton,  $\phi_{60}$ , is located in the steep region beyond the maximum of the potential. The value of the spectral index associated with this choice of  $\beta$  is  $n \simeq 1.29$  (a 3% below the first-order result  $n \simeq 1.34$ ). This value in excess of unity for the scales under consideration yields a spectrum with less power on large scales (compared with a Harrison-Zel'dovich one) making the long wave length gravitational wave contribution to the estimated quadrupole subdominant. The slow-roll parameters for this scale are  $\epsilon = 0.16$ ,  $\eta = 0.37$  and  $\xi = 1.01$  [41]. While the first two are smaller than one,  $\xi$  is not and so cannot be considered an

expansion variable on the same footing as  $\epsilon$  and  $\eta$ . Terms proportional to  $\xi$  (and therefore non negligible) in Eq.(6) are those not included in the second-order analysis done for the first time in Ref.[23]. Taking the *COBE* normalization we finally get  $\mathcal{S} \simeq -4.4 \times 10^{-5}$ .

Let us now consider  $\beta = 7$ . Now the potential falls to zero as a power-law much more rapidly than in the previous case. If we take  $n \simeq 0.8$  we see from Fig. 2 that  $d \simeq 6.05$  ( $\phi_{60} \simeq 2.65$ ) corresponding to the slow-rolling region of the potential [42]. For the parameters we find the following values:  $\epsilon = 0.13$ ,  $\eta = 0.17$  and  $\xi = 0.27$ . From these we see that the first-order result ( $n \simeq 0.86$ ) is 7 % above the full second-order one. In this case we get  $\mathcal{S} \simeq 0.50\mathcal{Q}$ . Now gravitational waves contribute substantially to the detected quadrupole. Actually we have  $\mathcal{Q}_T^2/\mathcal{Q}_S^2 \simeq 1.99$  [43, 39, 21] (while we would have had  $\sim 1.89$  up to first order). Thus the estimated quadrupole should be multiplied by a factor  $(1 + 1.99)^{-1/2}$  to correctly account for the tensor mode contribution. Finally we get  $\mathcal{S} \simeq 7.5 \times 10^{-6}$ .

### 3.3 Polynomial Potential

We are interested in considering a potential of the form

$$V(\phi) = A\left(\frac{1}{4}\phi^4 + \frac{\alpha}{3}\phi^3 + \frac{\beta}{8}\phi^2\right) + V_0 \quad (9)$$

where for convenience  $\phi$  is written in Planck mass units and  $A$ ,  $\alpha$  and  $\beta$  are dimensionless parameters. Translation invariance allows us to omit the linear  $\phi$  contribution to  $V$ . A detailed analysis of a potential of the form (9) was done by Hodges et al. [6]. Parameter space diagrams were constructed and regions where non-scale invariance was expected were isolated. Here we will just summarize what is necessary for our study.

Taking  $\beta > 8\alpha^2/9$  ensures that  $\phi = 0$  is the global minimum and therefore  $V_0 = 0$ . If we further require  $\beta > \alpha^2$  then no false vacua are present. Scalar curvature perturbations are conveniently expressed in terms of the gauge-invariant variable  $\zeta$  [44, 43]. The power spectrum associated with it, assuming slow-roll evolution of the scalar field in the relevant region of the potential, is given by  $P_\zeta^{1/2}(k) \propto H^2/\dot{\phi}$  evaluated at horizon crossing time. Equivalently  $P_\zeta^{1/2}(k) \propto V^{3/2}/(m_P^3 V')$  which suggests that regions of the parameter space

where the slope of the potential goes through a minimum or a maximum will be of interest as far as broken scale invariance is concerned. The presence of these extrema in  $V'$  is guaranteed by taking  $\alpha^2 < \beta < 4\alpha^2/3$ , with  $\alpha < 0$  in order for the scalar field to roll down the potential from the right. We will thus concentrate our study in the vicinity of an inflection point  $\phi_f$ , i.e. near the curve  $\beta = \alpha^2$  (but still  $\beta > \alpha^2$ ). In this limit we have  $\phi_f \simeq -\frac{\alpha}{2}$  and the number of e-foldings taking place in the region of approximately constant slope about  $\phi_f$  is given by  $N = -\pi\alpha^3/\sqrt{3(\beta - \alpha^2)}$  [6]. Fixing  $N = 60$  as the number required for achieving sufficient inflation, the interesting parameters ought to lie close to the curve  $\beta \simeq \alpha^2 + \pi^2\alpha^6/(3N^2) \simeq \alpha^2 + y\alpha^6$  with  $y \simeq 9.1 \times 10^{-4}$ .

We find  $X_{60}^2 - 4X'_{60} = 192 y (9y - \alpha^{-4})/[\alpha^2(6y + \alpha^{-4})^2]$  where, again, we are normalizing the field in Planck mass units,  $y \simeq 9.1 \times 10^{-4}$  and we have evaluated the field for  $\phi_{60} = \phi_f \simeq -\frac{\alpha}{2}$ .

Variation of  $\alpha$  in the allowed range results in both positive and negative values for  $\mathcal{S}$ . Clearly  $\beta > 10\alpha^2/9$  yields  $\mathcal{S} > 0$ . Although in this region of the parameter space the value of  $\alpha$  that makes  $\mathcal{S}$  maximal corresponds to  $\alpha = -5.44$ , this value conflicts with the requirement  $\beta < 4\alpha^2/3$  for the existence of an inflection point. We take instead  $\alpha = -3.69$  which makes  $\mathcal{S}$  the largest possible one and at the same time agrees within a few percent with approximating  $\phi_f \simeq -\frac{\alpha}{2}$ . Then  $\beta \simeq \alpha^2 + y\alpha^6 \simeq 15.92$ . This guarantees we are effectively exploring the neighbourhood of the curve  $\beta = \alpha^2$  in parameter space. A plot of the potential for these particular values is given in Fig. 4. The slow-roll parameters in this case have values:  $\epsilon = 5.94 \times 10^{-3}$ ,  $\eta = 5.84 \times 10^{-3}$  and  $\xi = 1.32 \times 10^{-1}$ . By using Eq.(6) we get  $n \simeq 0.99$ .

Let us consider now the case where  $\alpha^2 < \beta < 10\alpha^2/9$ . This choice of potential parameters leads to  $\mathcal{S} < 0$ . The non-Gaussian signal gets maximized for  $\alpha = -2.24$  and  $\beta = 5.13$ . Fig. 5 shows the potential in this case. Note the resemblance between this form of the potential and that of the hybrid [45] model  $V_0 + m^2\phi^2/2$  for  $V_0$  dominating and in the vicinity of the flat region. In that case the same sign of the dimensionless skewness [10] and blue spectra

[46],  $n > 1$ , were predicted. For this value of  $\alpha$  we get:  $\epsilon = 9.33 \times 10^{-4}$ ,  $\eta = 6.76 \times 10^{-3}$  and  $\xi = 2.76$ . We get  $n \simeq 1.01$ .

We can now use Eq.(4) for the quadrupole to find the overall normalization constant  $A$  of the potential. Bennett et al. [13] get a best fit  $Q_{rms-PS} = (17.6 \pm 1.5)\mu K$  for  $n$  fixed to one, as it is our present case to a very good approximation. Thus, for  $\alpha = -3.69$  we get  $A = 2.87 \times 10^{-12}$  and  $\mathcal{S} \simeq 1.1 \times 10^{-6}$  while for  $\alpha = -2.24$  we get  $A = 5.88 \times 10^{-12}$  and  $\mathcal{S} \simeq -1.9 \times 10^{-6}$ .

The above two values for  $n$  show that the departure from scale invariance is actually very small (in fact, this is because we are exploring a very narrow range of scales). Note also the relatively large value of  $\xi$  in the last case compared with  $\epsilon$  and  $\eta$ . This tells us that the terms proportional to  $\xi$  are non negligible in general. Also the rather small amplitudes for  $\mathcal{S}$  agree with previous numerical analyses [6] in which adherence to the correct level of anisotropies in the CMB radiation under the simplest assumptions of inflation, like slow-rolling down with potential (9), practically precludes any observable non-Gaussian signal.

## 4 Conclusions

In the present paper we have presented explicit calculations (in the frame of some well motivated inflationary models) of the dimensionless skewness  $\mathcal{S}$  predicted for the large angular scale temperature anisotropies in the CMB radiation as well as the evaluation of the primordial spectral index of the density perturbations originating these anisotropies. These computations were performed to full second order in perturbation theory. In all three models (even in the case of the polynomial inflaton potential where more parameters were at our disposal) very low values for the non-Gaussian signal were obtained.

In fact, the explicit values for  $\mathcal{S}$  were found generically much smaller than the dimensionless *rms* skewness calculated from an underlying Gaussian density field and are therefore hidden by this theoretical noise, making experimental detection impossible.

One may try to resort to many-field models in the hope of shifting the non-zero  $\mathcal{S}$  window

to larger values. In this respect a potential of the form  $V(\phi_i) \sim \exp(\sum_i \lambda_i \phi_i)$  is likely to do well the job [9]. In this case the resolution of a set of coupled Langevin-type equations for the coarse-grained fields (suitably smoothed over a scale larger than the Hubble radius) should be faced. In contrast to previous many-field analyses where one of the fields was assumed to dominate at a certain stage, in our case (for the aim of computing the three-point function) we need to make a second-order perturbative expansion in  $\delta\phi_i$  around the classical solutions  $\phi_i^{class}$  but keeping  $V(\phi_i)$  fully dependent on all the fields. Non-Gaussian fluctuations can indeed be generated within this model and thus the prospect of getting a non-negligible value for the dimensionless skewness should be tested. This is the subject of our current research.

**Acknowledgements:** The author is grateful to S. Matarrese for valuable discussions. He also thanks A. Conti for his kind help in the numerical calculations. This work was supported by the Italian MURST.

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## Figure Captions

**Fig. 1** : Dimensionless skewness  $\mathcal{S}$  (in units of the quadrupole  $\mathcal{Q}$ , see Eq.(8)) as a function of  $d \equiv \phi^2 - \beta^2/16\pi$ , the value of the squared of the field beyond the minimum allowed, for different choices of  $\beta$ . Curves from bottom to top correspond to  $\beta$  from 1 to 8 (left panel).

**Fig. 2** : Spectral index (as calculated from Eq.(6)) as a function of  $d$ . Now curves from top to bottom are those corresponding to  $\beta$  from 1 to 8 (right panel).

**Fig. 3** : Inflaton potential  $V(\phi)$  for the parameter choice  $\beta = 1.2$  ( $f = 0.77$ ). The field is taken in units of  $m_P$ , while the potential is normalized in units of  $(10^{-1}m_PA^{1/f})^2$ .

**Fig. 4** : Polynomial potential  $V(\phi)$  as a function of the scalar field ( $\phi$  is taken in Planck mass units). The overall normalization parameter  $A$  is taken to be one; for  $\alpha = -3.69$  and  $\beta = 15.92$  (left panel);

**Fig. 5** : Polynomial potential  $V(\phi)$  as a function of the scalar field ( $\phi$  is taken in Planck mass units). The overall normalization parameter  $A$  is taken to be one; for  $\alpha = -2.24$  and  $\beta = 5.13$  (right panel).

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